King Fahd University of Petroleum & Minerals
College of Computer Sciences & Engineering
Department of Information and Computer Science

ICS 253: Discrete Structures I
Final Exam – 131
120 Minutes

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Question	Max	Earned
1	21	
2	9	
3	7	
4	7	
5	24	
6	18	
7	7	
8	7	
Total	**100**	

Wednesday, January 1, 2014

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Question 1: [21 Points] Indicate whether the given sentence is true or false. In the answer column write either ✓ for "true" or × for "false".

Statement	Answer
1. The negation of the proposition "Ahmad's PC runs Linux" is "Ahmad's PC runs Windows".	×
2. The contrapositive of the conditional statement "The home team wins whenever it is raining?" is "If the home team does not win, then it is not raining."	✓
3. Logical connectives are used extensively in searches of large collections of information, such as indexes of Web pages.	✓
4. $\neg (p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.	✓
5. If $P(x)$ is the statement "$x + 1 > x$" where the domain consists of all real numbers, then $\forall x\, P(x)$ *is false.*	×
6. The quantifiers ∀ and ∃ have higher precedence than all logical operators from propositional calculus.	✓
7. $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent.	✓
8. the statement $\forall x \exists y(x + y = 0)$ says that every real number has an additive inverse.	✓
9. An *onto* function $f: A \rightarrow B$ maps the A over a piece of the set B, not over the *entirety* of it.	×
10. If x and y are integers and both xy and $x + y$ are even, then both x and y are odd.	×
11. The set of all positive integers less than 100 can be denoted by $\{1, 4, 5, \ldots, 99\}$.	×
12. If $A = \{1, 2\}$, then $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.	✓
13. If $A_1 = \{0, 2, 4, 6, 8\}$, $A_2 = \{0, 1, 2, 3, 4\}$, and $A_3 = \{0, 3, 6, 9\}$, then $\bigcap_{i=1}^{3} A_i = \{0\}$.	✓
14. The function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.	×
15. A function is not invertible if it is not a one-to-one correspondence.	✓
16. $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$	✓
17. The formula for the sequence 1, 1/2, 1/4, 1/8, 1/16, … is $a_n = 1/4^n$, $n = 0, 1, 2, \ldots$	×
18. $\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2$	✓
19. If A and B are countable sets, then $A \cup B$ is also countable.	✓
20. Mathematical induction can be used to prove mathematical statements that assert a property is true for all positive integers such as "for every positive integer n: $n! \leq n^n$."	✓
21. There are 1000 positive integers devisable by 9 between 1000 and 9999 inclusive.	✓

Question 2: [9 Points] Fill in the first column of the table below by writing the number of the ***most proper*** text from the **3rd column** that is related to the text in the **2nd column** (only one number per entry):

Number	**2nd Column**	**3rd Column**
1	If k is a positive integer and k +1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.	1. The Pigeonhole Principle
3	The number of r-combinations of a set with n distinct elements ($C(n, r)$ *or* $\binom{n}{r}$)	2. Application of Probability Theory
4	Determining the number of moves in the Tower of Hanoi puzzle and counting bit strings with certain properties.	3. Binomial Coefficients
7	The inductive step shows that if $P(j)$ is true for all positive integers not exceeding k, then $P(k+1)$ is true.	4. Applications of Recurrence Relations
6	The inductive step shows that if the inductive hypothesis $P(k)$ is true, then $P(k + 1)$ is also true.	5. Rules of Inference
2	Extensively applied in the study of genetics, where it can be used to help understand the inheritance of traits.	6. Mathematical Induction
5	Basic tools for establishing the truth of statements. They are templates for constructing valid arguments.	7. Strong Induction
8	$\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$	8. Nested Quantifiers
9	Collection of Objects.	9. Sets

Question 3: [7 Points] Strong Induction

Using strong induction, prove that for each positive integer n, the n^{th} Fibonacci number f_n is less than $(7/4)^n$. Note that Fibonacci numbers are defined as follows: $f_0=1$, $f_1=1$ and $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$.

Basis: $f_1=1 < (7/4)^1$ and $(7/4)^2 = (49/16) > (32/16) = 2 = 1 + 1 = f_1 + f_2$.

Inductive Step: Assume that the result holds for n, $2 \leq n \leq k$, $k, n \in Z$, i.e., $f_n < (7/4)^n$.

To show that it holds for $k + 1$, i.e., $f_{k+1} < (7/4)^{k+1}$

$f_{k+1} = f_k + f_{k-1} < (7/4)^k + (7/4)^{k-1}$ (by Induction Hypothesis)

$= (7/4)^{k-1} [7/4 + 1] = (7/4)^{k-1} (11/4)$

$= (7/4)^{k-1} (44/16) < (7/4)^{k-1} (49/16)$

$= (7/4)^{k-1} (7/4)^2 = (7/4)^{k+1}$

Question 4: [7 Points] Structural Induction

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a, b) \in S$, then $(a, b + 1) \in S$, $(a + 1, b + 1) \in S$, and $(a + 2, b + 1) \in S$.

Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in S$.

Basis: $0 \leq (2)(0) = 0$.

Structural Induction Step: If this holds for (a, b) $\in S$, i.e., $a \leq 2b$, then

$$a \leq 2b \rightarrow a \leq 2b + 2 = 2(b + 1)$$

$$a \leq 2b \rightarrow a + 1 \leq 2b + 2 = 2(b + 1)$$

$$a \leq 2b \rightarrow a + 2 \leq 2b + 2 = 2(b + 1)$$

Question 5: [24 Points] Counting and Applications

a) **[5 points]** How many positive integers not exceeding 200 are divisible by 4 or 5?

$A = \{x | 1 \le x \le 200\ \land 4 \text{ divides } x\}, |A| = \frac{200}{4} = 50$

$B = \{x | 1 \le x \le 200\ \land 5 \text{ divides } x\}, |B| = \frac{200}{5} = 40$

$|A\ \cap B| = |A| + |B| - |A\ \cap b| = 50 + 40 - |\{x|1 \le x \le 200\ \land 20 \text{ divides } x\}| = 90 - 10 = \mathbf{80}$

b) **[5 points]** What is the coefficient of $x^{13}y^{37}$ in the expansion of $(3y - x)^{50}$?

$$\binom{50}{13}(-1)^{13}(3)^{37} = -\binom{50}{13}3^{37}$$

c) **[7 points]** How many bit strings of length 12 have exactly four 1s such that all the 1s are separated by 0s (so no two 1s are adjacent)?

In each of the bit strings, 1s have to be separated by 0s.

There are exactly eight 0s in each bit string, and nine positions in between the 0s to put the 1s.

_ 0 _ 0_ 0 _ 0_ 0 _ 0_ 0 _ 0 _

We need to select four out of these nine positions to put the four 1s. Answer: **C(9, 4)**

d) **[7 points]** Recall that a set of 52 playing cards is divided equally into 4 suits. Use the pigeonhole principle to find an expression for the least number of cards required to ensure that at least *x* cards are of the same suit, where $1 \leq x \leq 13$.

Let *N* be the number of required cards.

By the pigeonhole principle, and since we have 4 suits, $\left\lceil \frac{N}{4} \right\rceil = x$.

So, $x - 1 < \frac{N}{4} \leq x$

$4(x - 1) < N \leq 4x$

$\boldsymbol{N \geq 4(x-1) + 1 = 4x - 3}$.

Question 6: [18 Points] Discrete Probability

a) **[12 points]** A bit string of length six is picked at random such that all bit strings are equally likely. Consider the following events:

E_1: the bit string begins with 1;
E_2: the bit string ends with 1;
E_3: the bit string has exactly three 1s.

i. **[4 points]** Find $p(E_1 \mid E_3)$.

$p(E_1|E_3) = \frac{p(E_1 \cap E_3)}{p(E_3)}$

$p(E_3) = \frac{|E_3|}{|S|} = \frac{C(6,3)}{2^6} = \frac{20}{64} = \frac{5}{16}$

$p(E_1 \cap E_3) = \frac{|E_1 \cap E_3|}{|S|} = \frac{C(5,2)}{2^6} = \frac{10}{64} = \frac{5}{32}$

$p(E_1|E_3) = \frac{p(E_1 \cap E_3)}{p(E_3)} = \mathbf{\frac{1}{2}}$

ii. **[2 points]** Are E_1 and E_3 independent? Justify.

Yes, since $p(E_1) = \frac{|E_1|}{|S|} = \frac{2^5}{2^6} = \frac{1}{2}$ is equal to $p(E_1|E_3)$.

iii. **[6 points]** Are E_1 and E_2 independent? Justify.

$p(E_2) = \frac{|E_2|}{|S|} = \frac{2^5}{2^6} = \frac{1}{2}$

$p(E_1 \cap E_2) = \frac{|E_1 \cap E_2|}{|S|} = \frac{2^5}{2^6} = \frac{1}{4}$

Yes, since $p(E_1 \cap E_2) = p(E_1)P(E_2)$.

b) **[6 points]** An unfair coin is flipped ten times, where *p*(heads) = 3/4. Find the following:
p(at least 3 heads appear | at least 1 head appears).

$p(\text{at least 3H}) = 1 - (p(0\text{H}) + p(1H) + p(2\text{H}))$

$p(\text{at least 1H}) = 1 - p(0\text{H})$

$p(\text{at least 3H} \;\cap\; \text{at least 1H}) = p(\text{at least 3H})$, since $(\text{at least 3H}) \subseteq (\text{at least 1H})$

$$p(\text{at least 3H} \mid \text{at least 1H}) = \frac{p(\text{at least 3H} \;\cap\; \text{at least 1H})}{p(\text{at least 1H})} = \frac{1-(p(0\text{H})+p(1H)+p(2\text{H}))}{1-p(0\text{H})}$$

where

$$p(0\text{H}) = \binom{10}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^{10} = \left(\frac{1}{4}\right)^{10}$$

$$p(1\text{H}) = \binom{10}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^{9}$$

$$p(0\text{H}) = \binom{10}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{8}$$

Question 7: [7 Points] Applications of Recurrence Relations

a) **[4 points]** Find a recurrence relation for the number of ternary strings of length *n* that contain two consecutive 0s.
Note: Ternary strings are strings with characters from {0,1,2}.

$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$

b) **[1 point]** What are the initial conditions?

$a_0 = 0$ **and** $a_1 = 0$ **OR**

$a_1 = 0$ **and** $a_2 = 1$

c) **[2 points]** How many ternary strings of length five contain two consecutive 0s?

$a_3 = 2a_2 + 2a_1 + 3^1 = 2 + 0 + 3 = 5$

$a_4 = 2a_3 + 2a_2 + 3^2 = 10 + 2 + 9 = 21$

$a_5 = 2a_4 + 2a_3 + 3^3 = 42 + 10 + 27 = 79$

Question 8: [7 Points] Solving Linear Recurrence Relations

Solve the following linear recurrence relation, together with the initial conditions given.

$$a_n = -6a_{n-1} - 9a_{n-2} \text{ for } n \geq 2 \text{ and } a_0 = 3 \text{ and } a_1 = -3$$

Characteristic Equation: $x^2 + 6x + 9 = 0$

$(x + 3)^2 = 0$

$x = -3$

$\therefore a_n = \alpha_1(-3)^n + \alpha_2 n(-3)^n$

Using the initial conditions

$a_0 = 3 = \alpha_1(-3)^0 + \alpha_2.0.(-3)^0 = \alpha_1$

$a_1 = -3 = \alpha_1(-3)^1 + \alpha_2.1.(-3)^1 = -3\alpha_1 - 3\alpha_2 = -9 - 3\alpha_2$

$\alpha_2 = -2$

$\therefore a_n = 3(-3)^n - 2n(-3)^n$

Some Useful Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad , \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad , \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1} \text{ where } a \neq 1 \quad , \quad \sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \text{ where } |a|<1, \ \sum_{i=1}^{\infty} i a^{i-1} = \frac{1}{(1-a)^2} \text{ where } |a|<1$$

$p \to (p \vee q)$	Addition	$[\neg q \wedge (p \to q)] \to \neg p$	Modus Tollens
$(p \wedge q) \to p$	Simplification	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$((p) \wedge (q)) \to (p \wedge q)$	Conjunction	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
$[p \wedge (p \to q)] \to q$	Modus Ponens	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws